Math 3450 - Homework # 3
Well-Defined Operations

1. Show that the operation $\overline{a} \oplus \overline{b} = \overline{a}^2 + \overline{b}^2$ is a well-defined operation for $\mathbb{Z}_n$. Here $\overline{a}^2$ means $\overline{a} \cdot \overline{a}$. For example, in $\mathbb{Z}_4$ we have that

$$\overline{2} \oplus \overline{3} = \overline{2} \cdot \overline{2} + \overline{3} \cdot \overline{3} = \overline{4} + \overline{9} = \overline{1}.$$

2. Given two integers a and b, let $\min(a, b)$ denote the minimum (smaller) of a and b. Let n be an integer with $n \geq 2$. Is the operation $\overline{a} \oplus \overline{b} = \overline{\min(a, b)}$ a well-defined operation on $\mathbb{Z}_n$?

3. (a) Show that the operation $\frac{a}{b} \oplus \frac{c}{d} = \frac{ad}{bc}$ is not a well-defined operation on $\mathbb{Q}$.

   (b) Is the operation well-defined on $\mathbb{Q} - \{0\}$?

4. Is the operation $\overline{a} \oplus \overline{b} = \overline{a^b}$ a well-defined operation on $\mathbb{Z}_n$?

5. (Constructing the rational numbers from the integers) Let $S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Define the relation $\sim$ on S where $(a, b) \sim (c, d)$ if and only if $ad = bc$. In the last homework you showed that this is an equivalence relation on S.

   (a) Define the operation $\overline{(a, b)} \oplus \overline{(c, d)} = \overline{(ad + bc, bd)}$. Prove that $\oplus$ is well-defined on the set of equivalence classes.

   (b) Define the operation $\overline{(a, b)} \odot \overline{(c, d)} = \overline{(ac, bd)}$. Prove that $\odot$ is well-defined on the set of equivalence classes.

6. (Constructing the integers from the natural numbers) Let $S = \mathbb{N} \times \mathbb{N}$. Define the relation $\sim$ on S where $(a, b) \sim (c, d)$ if and only if $a+d = b+c$. In the last homework you showed that this is an equivalence relation on S.

   (a) Define the operation $\overline{(a, b)} \oplus \overline{(c, d)} = \overline{(a + c, b + d)}$. Prove that $\oplus$ is well-defined on the set of equivalence classes.